

Characterizations of the (b, c) -inverse in a ring

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Abstract: Let R be a ring and $b, c \in R$. In this paper, we give some characterizations of the (b, c) -inverse, in terms of the direct sum decomposition, the annihilator and the invertible elements. Moreover, elements with equal (b, c) -idempotents related to their (b, c) -inverses are characterized, and the reverse order rule for the (b, c) -inverse is considered.

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1. INTRODUCTION

Moore-Penrose inverse, Drazin inverse and group inverse, as for the classical generalized inverses, are special types of outer inverses. In [8], Drazin introduced a new class of outer inverse in a semigroup and called it (b, c) -inverse.

Definition 1.1. *Let R be an associative ring and let $b, c \in R$. An element $a \in R$ is (b, c) -invertible if there exists $y \in R$ such that*

$$y \in (bRy) \cap (yRc), \quad yab = b, \quad cay = c.$$

If such y exists, it is unique and is denoted by $a^{\parallel(b,c)}$.

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From [8], we know that the Moore-Penrose inverse of a , with respect to an involution $*$ of R , is the (a^*, a^*) -inverse of a , the Drazin inverse of a is the (a^j, a^j) -inverse of a for some $j \in \mathbb{N}$, in particular, the group inverse of a is the (a, a) -inverse of a .

Given two idempotents e and f , Drazin introduced the Bott-Duffin (e, f) -inverse in [8], which can be considered as a particular cases of the (b, c) -inverse. In 2014, Kantún-Montiel introduced the image-kernel (p, q) -inverse for two idempotents p and q , and pointed out that an element a is image-kernel (p, q) -invertible if and only if it is Bott-Duffin $(p, 1 - q)$ -invertible [9, Proposition 3.4]. In [11], elements with equal idempotents related to their image-kernel (p, q) -inverses are characterized in terms of classical invertibility. The topics of research on the image-kernel (p, q) -inverse and the Bott-Duffin (e, f) -inverse attract wide interest (see [2–4, 6–9, 11]).

This article is motivated by the papers [8, 11]. In [8], as a generalization of (b, c) -inverse, hybrid (b, c) -inverse and annihilator (b, c) -inverse were introduced. In section 3, it is shown that if the (b, c) -inverse of a exists, then both b and c are regular. Further, under the natural hypothesis of both b and c regular, some characterizations of the (b, c) -inverse are obtained in terms of the direct sum decomposition, the annihilator and the invertible elements. In particular, we will prove that (b, c) -inverse, hybrid (b, c) -inverse and annihilator (b, c) -inverse are coincident. Some results of the image-kernel (p, q) -inverse in [11] are generalized.

If a has a (b, c) -inverse, then both $a^{\parallel(b, c)}a$ and $aa^{\parallel(b, c)}$ are idempotents. These will be referred as to the (b, c) -idempotents associated with a . In [5], Castro-González, Koliha and Wei characterized matrices with the same spectral idempotents corresponding to the Drazin inverses of these matrices. Koliha and Patrício [10] extend the results to the ring case. A similar question for the Moore-Penrose inverse was considered in [12]. In [11], Mosić gave some characterizations of elements which have the same idempotents related to their image-kernel (p, q) -inverses. It is of interest to know whether two elements in the ring have equal (b, c) -idempotents. In section 4, some characterizations of those elements with equal (b, c) -idempotents are given. Moreover, the reverse order rule for the (b, c) -inverse is considered.

2. PRELIMINARIES

Let R be an associative ring with unit 1. Let $a \in R$. Recall a is a regular element if there exists $x \in R$ such that $a = axa$. In this case, the element x is called an inner inverse for a and we will denote it by a^- . If the equation $x = xax$ is satisfied, then we say that a is outer generalized invertible and x is called an outer inverse for a . An element x that is both inner and outer inverse of a and commutes with a , when it exist, must be unique and is called the group inverse of a , denoted by $a^\#$. From now on, $E(R)$ and $R^\#$ stand for the set of all idempotents and the set of all group invertible elements in R . For the sake of convenience, we introduce some necessary notations.

For an element $a \in R$ and $X \subseteq R$, we define

$$aR := \{ax : x \in R\}, \quad Ra := \{xa : x \in R\};$$

$$l(X) := \{y \in R : yx = 0 \text{ for any } x \in X\}, \quad r(X) := \{y \in R : xy = 0 \text{ for any } x \in X\}.$$

In particular,

$$l(a) := \{y \in R : ya = 0\}, \quad r(a) := \{y \in R : ay = 0\},$$

$$rl(a) = \{y : xy = 0, x \in l(a)\} \text{ and } lr(a) = \{y : yx = 0, x \in r(a)\}.$$

Let $p, q \in E(R)$. An element $a \in R$ has an image-kernel (p, q) -inverse [9, 11] if there exists an element $c \in R$ satisfying

$$cac = c, \quad caR = pR, \quad (1 - ac)R = qR.$$

The image-kernel (p, q) -inverse is unique if it exists, and it will be denoted by a^\times . A generalization of the original Bott-Duffin inverse [1] was given in [8]: let $e, f \in E(R)$, an element $a \in R$ is Bott-Duffin (e, f) -invertible if there exist $y \in R$ such that $y = ey = yf$, $yae = e$ and $fay = f$. When $e = f$, the element y , if any, is given by $y = e(ae + 1 - e)^{-1}$ as for the original Bott-Duffin inverse.

The above mentioned generalized inverses are particular cases of the (b, c) -inverse where b and c have the property of being both idempotents. Hence, the research of (b, c) -inverse has important significance to the development of the generalized inverse theory.

For the future reference we state two known results.

Lemma 2.1. [8, Theorem 2.2] *For any given $a, b, c \in R$, there exists the (b, c) -inverse y of a if and only if $Rb = Rt$ and $cR = tR$, where $t = cab$.*

Lemma 2.2. [8, Proposition 6.1] *For any given $a, b, c \in R$, y is the (b, c) -inverse of a if and only if $yay = y$, $yR = bR$ and $Ry = Rc$.*

3. SOME CHARACTERIZATIONS OF THE EXISTENCE OF (b, c) -INVERSE

Firstly, we will give some lemmas which will be used in the sequel.

Lemma 3.1. *Let $a, y \in R$ such that y is an outer inverse of a . Then*

- (i) $r(a) \cap yR = \{0\}$.
- (ii) $l(a) \cap Ry = \{0\}$.
- (iii) $Ray = Ry$.
- (iv) $yaR = yR$.

Proof. (i). Let $x \in r(a) \cap yR$. Then $ax = 0$ and there exists $g \in R$ such that $x = yg$. This gives that $ayg = 0$ and, thus, $yayg = yg = 0$. Therefore, $x = 0$.

(ii). Let $x \in l(a) \cap Ry$. Then $xa = 0$ and there exists $h \in R$ such that $x = hy$. It leads to $hya = 0$. Then $hyay = hy = 0$ and, thus, $x = 0$.

(iii) and (iv). From $yay = y$ it follows that $yaR = yR$ and $Ry = Ray$. □

Lemma 3.2. *Let $a \in R$ be regular and $b \in R$. Then*

- (i) b is regular in case $Ra = Rb$.
- (ii) $rl(a) = aR$ and $lr(a) = Ra$.

Proof. (i). Since $Ra = Rb$, there exist some $g, h \in R$ such that $a = gb$ and $b = ha$. Hence, using that a is regular, one can see $b = (ha)a^-a = ba^-a = ba^-gb$, which means that b is regular.

(ii). It is easy to check that $aR \subseteq rl(a)$. Note that $l(a) = l(aa^-) = R(1 - aa^-)$. For any $x \in rl(a)$, one can get $R(1 - aa^-)x = l(a)x = 0$. This gives $x = aa^-x \in aR$ and $rl(a) = aR$. Similar considerations apply to prove that $lr(a) = Ra$. □

Proposition 3.3. *If a has a (b, c) -inverse, then b , c and $t = cab$ are all of them regular.*

Proof. Let y be the (b, c) -inverse of a . In view of Definition 1.1, one can see $b = yab \in (bRy)ab \subseteq bRb$. This gives that b is regular. In the same manner one can obtain that c is regular. Now, on account of Lemma 2.1, we have $Rb = Rt$ and $cR = tR$ since the (b, c) -inverse of a exists. From Lemma 3.2, we conclude that t is regular. □

In what follows, we will give necessary and sufficient conditions for the existence of the (b, c) -inverse when $t = cab$ is regular.

Theorem 3.4. *Let $a, b, c \in R$. If $t = cab$ is regular, then the following statements are equivalent:*

- (i) a has a (b, c) -inverse.
- (ii) $r(a) \cap bR = \{0\}$ and $R = abR \oplus r(c)$.
- (iii) $r(t) = r(b)$ and $tR = cR$.
- (iv) $l(t) = l(c)$ and $Rt = Rb$.
- (v) $l(t) = l(c)$ and $r(t) = r(b)$.

Proof. (i) \Rightarrow (ii) Suppose that y is the (b, c) -inverse of a . By Lemma 2.2, $yay = y$, $yR = bR$ and $Ry = Rc$. By Lemma 3.1 (i), one can see $r(a) \cap yR = \{0\}$, it follows that $r(a) \cap bR = \{0\}$. Since $ay \in E(R)$, we have the decomposition $R = ayR \oplus r(ay)$. From $yR = bR$ we obtain $ayR = abR$. By Lemma 3.1 (iii) and $Ry = Rc$, then $Ray = Rc$ and hence $r(ay) = r(c)$. Consequently, we have $R = abR \oplus r(c)$.

(ii) \Rightarrow (iii). It is clear that $r(b) \subseteq r(t)$. For any $x \in r(t)$, we have $tx = cabx = 0$. This means that $abx \in r(c)$. Using that $r(c) \cap abR = \{0\}$ we conclude that $abx = 0$. Then $bx \in r(a) \cap bR = \{0\}$. This implies that $bx = 0$ and, thus, $x \in r(b)$. Therefore $r(t) = r(b)$.

It is clear that $tR \subseteq cR$. Since $R = abR \oplus r(c)$, we can write $1 = abg + h$ where $g \in R$ and $h \in r(c)$. Premultiplying by c gives $c = cabg \in tR$, ensuring that $cR = tR$.

(iii) \Rightarrow (iv). Since $tR = cR$, we have $l(c) = l(t)$. It is clear the $Rt \subseteq Rb$. Using that t is regular and $r(t) = r(b)$ we obtain that $b(1 - t^-t) = 0$. Then $b = bt^-t$. Consequently, $Rt = Rb$.

(iv) \Rightarrow (v). It is clear.

(v) \Rightarrow (i). Since $r(t) = r(b)$ and t is regular we can prove that $Rt = Rb$ as in the proof of (iii) \Rightarrow (iv). Similarly, from $l(t) = l(c)$ and the fact that t is regular we get $tR = cR$. On account of Lemma 2.1 we conclude that a has a (b, c) -inverse. \square

In Theorem 3.4, the implications (i) \Rightarrow (ii) and (ii) \Rightarrow (iii) are valid even if t is not regular. However, we will give a counterexample to show that (iii) does not imply (iv) in general when t is not regular.

Example 3.5. Set $R = \mathbb{Z}$, $a = b = 1$ and $c = 2$. Clearly, $tR = cR$ and $r(t) = r(b)$, but $Rb \neq Rt$.

When we replace the hypothesis that t is regular in Theorems 3.4 by the condition that both b and c are regular, we obtain the following result.

Theorem 3.6. Let $a, b, c \in R$. If both b and c are regular, then the statements (i)-(iv) in Theorem 3.4 are equivalent.

Proof. We note that in item (iii) condition $tR = cR$ together with c is regular implies that t is regular, in item (iv) $Rt = Rb$ together with b is regular implies that t is regular. \square

Remark 3.7. The statements $(v) \Rightarrow (i)$ in Theorem 3.4 is not true, when b and c are regular. For example, set $R = \mathbb{Z}$, $b = c = 1$ and $a = 2$. Then b and c are regular. It is easy to check that $l(t) = l(c)$ and $r(t) = r(b)$, but $t = 2$ is not regular. Then a is not (b, c) -invertible by Proposition 3.3.

As a generalization of (b, c) -inverse, hybrid (b, c) -inverse and annihilator (b, c) -inverse were introduced in [8].

Definition 3.8. Let $a, b, c, y \in R$. We say that y is a hybrid (b, c) -inverse of a if

$$yay = y, \quad yR = bR, \quad r(y) = r(c).$$

Definition 3.9. Let $a, b, c, y \in R$. We say that y is a annihilator (b, c) -inverse of a if

$$yay = y, \quad l(y) = l(b), \quad r(y) = r(c).$$

In [8], Drazin pointed out that for any given $a, b, c \in R$,

$$(b, c)\text{-invertible} \Rightarrow \text{hybrid } (b, c)\text{-invertible} \Rightarrow \text{annihilator } (b, c)\text{-invertible}.$$

In what follows, we will prove that the three generalized inverses are coincident whenever $t = cab$ is regular.

Theorem 3.10. Let $a, b, c, y \in R$. If t is regular, then the following conditions are equivalent:

- (i) y is the (b, c) -inverse of a .
- (ii) y is the hybrid (b, c) -inverse of a .
- (iii) y is the annihilator (b, c) -inverse of a .

Proof. (i) \Rightarrow (ii) \Rightarrow (iii). These implications are clear.

(iii) \Rightarrow (i). By Definition 3.9, we have $1 - ay \in r(y) = r(c)$ and $1 - ya \in l(y) = l(b)$. This implies that $c = cay$ and $b = yab$. Next, we will prove that $r(t) = r(b)$ and $l(t) = l(c)$. Combining with Theorem 3.4 (v), then we can find that

$$a \text{ is annihilator } (b, c)\text{-invertible} \Rightarrow a \text{ is } (b, c)\text{-invertible}.$$

It is clear that $r(b) \subseteq r(t)$. Let $w \in r(t)$. Then $cabw = 0$ and hence $abw \in r(c) = r(y)$. This implies that $yabw = 0$. Then $bw = 0$ since $yab = b$. This shows $r(t) \subseteq r(b)$. Therefore, $r(t) = r(b)$. Similarly, we can prove that $l(c) = l(t)$. Since a has a (b, c) -inverse z , then a has the annihilator (b, c) -inverse z and by the uniqueness we have $z = y$. \square

Theorem 3.11. *Let $a, b, c \in R$. If both b and c are regular, then the statements (i)-(iii) in Theorem 3.10 are equivalent.*

Proof. We only need to prove that (iii) \Rightarrow (i). If y is the annihilator (b, c) -inverse of a , then $l(y) = l(b)$, this gives that $rl(y) = rl(b)$. Since b and y are regular, we have $rl(b) = bR$ and $rl(y) = yR$ by Lemma 3.2 (ii). This implies that $yR = bR$. Similarly, we can obtain that $Ry = Rc$. Thus, it follows that y is the (b, c) -inverse of a by Lemma 2.2. \square

The following lemma it is well known.

Lemma 3.12. *Let $a \in R$ and $e \in E(R)$. Then the following conditions are equivalent:*

- (i) $e \in eaeR \cap Reae$.
- (ii) $eae + 1 - e$ is invertible (or $ae + 1 - e$ is invertible).

Theorem 3.13. *Let $a, b, c, d \in R$ such that the (b, c) -inverse of a exists. Let $e = bb^-$ where b^- are fixed, but arbitrary inner inverses of b . Then the following statements are equivalent:*

- (i) d has a (b, c) -inverse.
- (ii) $e \in ea^{\parallel(b, c)}deR \cap Rea^{\parallel(b, c)}de$.
- (iii) $a^{\parallel(b, c)}de + 1 - e$ is invertible.

In this case,

$$(3.1) \quad d^{\parallel(b, c)} = (a^{\parallel(b, c)}de + 1 - e)^{-1}a^{\parallel(b, c)}.$$

Proof. Firstly, as $a^{\parallel(b,c)}$ exists we have $a^{\parallel(b,c)} \in bR \cap Rc$ by Lemma 2.2. Therefore

$$(3.2) \quad a^{\parallel(b,c)} = bb^- a^{\parallel(b,c)} = a^{\parallel(b,c)} c^- c.$$

From Definition 1.1 we have $b = a^{\parallel(b,c)} ab$. Combining with (3.2), we can write

$$(3.3) \quad b = ea^{\parallel(b,c)} c^- cab.$$

(i) \Rightarrow (ii). Suppose that $d^{\parallel(b,c)}$ exists. By Definition 1.1, we also have $c = cdd^{\parallel(b,c)}$. Substituting this into (3.3) yields

$$b = ea^{\parallel(b,c)} c^- (cdd^{\parallel(b,c)}) ab = ea^{\parallel(b,c)} dd^{\parallel(b,c)} ab.$$

Multiplying on the right by b^- we obtain $e = ea^{\parallel(b,c)} dd^{\parallel(b,c)} ae$. Since $d^{\parallel(b,c)} = ed^{\parallel(b,c)}$, which follows by interchanging $a^{\parallel(b,c)}$ and $d^{\parallel(b,c)}$ in (3.2), we get $e = ea^{\parallel(b,c)} ded^{\parallel(b,c)} ae$. This implies that $e \in ea^{\parallel(b,c)} deR$. Similarly, we can prove that $e \in Rea^{\parallel(b,c)} de$.

(ii) \Rightarrow (iii) See Lemma 3.12.

(iii) \Rightarrow (i) Firstly we note that $ea^{\parallel(b,c)} = a^{\parallel(b,c)}$ by (3.2). Set $x = ea^{\parallel(b,c)} de + 1 - e$. It is clear that $ex = xe$ and $ex^{-1} = x^{-1}e$. Write $y = x^{-1}a^{\parallel(b,c)}$. Next, we verify that y is the (b, c) -inverse of d .

Step 1. $ydy = y$. Indeed,

Using $a^{\parallel(b,c)} = ea^{\parallel(b,c)}$, we get

$$\begin{aligned} ydy &= x^{-1}a^{\parallel(b,c)} dx^{-1}a^{\parallel(b,c)} = x^{-1}ea^{\parallel(b,c)} dx^{-1}ea^{\parallel(b,c)} \\ &= x^{-1}(ea^{\parallel(b,c)} de + 1 - e)ex^{-1}a^{\parallel(b,c)} \\ &= x^{-1}ea^{\parallel(b,c)} = x^{-1}a^{\parallel(b,c)} = y. \end{aligned}$$

Step 2. $bR = yR$.

On account of $a^{\parallel(b,c)} = ea^{\parallel(b,c)}$ and $(1 - e)b = 0$, one can get

$$b = x^{-1}(ea^{\parallel(b,c)} de + 1 - e)b = x^{-1}ea^{\parallel(b,c)} deb = x^{-1}a^{\parallel(b,c)} deb = ydeb \in yR$$

Meanwhile, $y = x^{-1}a^{\parallel(b,c)} = x^{-1}ea^{\parallel(b,c)} = ex^{-1}a^{\parallel(b,c)} \in bR$. This guarantees $bR = yR$.

Step 3. $Rc = Ry$.

From Definition 1.1, we have $c = caa^{\parallel(b,c)}$. This leads to $c = caxx^{-1}a^{\parallel(b,c)} = caxy \in Ry$. On the other hand, from (3.2) we conclude that $y = x^{-1}a^{\parallel(b,c)} = x^{-1}a^{\parallel(b,c)} c^- c \in Rc$. It means that $Rc = Ry$. \square

Similarly, we can state the analogue of Theorem 3.13.

Theorem 3.14. *Let $a, b, c, d \in R$ such that the (b, c) -inverse of a exists. Let $f = c^-c$ where c^- are fixed, but arbitrary inner inverses of c . Then the following statements are equivalent:*

- (i) d has a (b, c) -inverse.
- (ii) $f \in fda^{\|(b,c)}fR \cap Rfda^{\|(b,c)}f$.
- (iii) $fda^{\|(b,c)} + 1 - f$ is invertible.

In this case,

$$(3.4) \quad d^{\|(b,c)} = a^{\|(b,c)}(fda^{\|(b,c)} + 1 - f)^{-1}.$$

Remark 3.15. In case that both $a^{\|(b,c)}$ and $d^{\|(b,c)}$ exist, from Theorem 3.13 and 3.14, it may be concluded that

$$(3.5) \quad \begin{aligned} (a^{\|(b,c)}de + 1 - e)^{-1} &= d^{\|(b,c)}ae + 1 - e; \\ (fda^{\|(b,c)} + 1 - f)^{-1} &= fad^{\|(b,c)} + 1 - f. \end{aligned}$$

Indeed, since $d^{\|(b,c)} = (a^{\|(b,c)}de + 1 - e)^{-1}a^{\|(b,c)}$, we have $(a^{\|(b,c)}de + 1 - e)d^{\|(b,c)} = a^{\|(b,c)}$. Hence,

$$(a^{\|(b,c)}de + 1 - e)(d^{\|(b,c)}ae + 1 - e) = a^{\|(b,c)}ae + 1 - e = 1,$$

where the last identity is due to the fact that $a^{\|(b,c)}ae = e$, because $b = a^{\|(b,c)}ab$. Interchanging the roles of a and d in Theorem 3.13 it follows that $(d^{\|(b,c)}ae + 1 - e)(a^{\|(b,c)}de + 1 - e) = 1$ and, in consequence, the first identity in (3.5) holds. The second identity in (3.5) can be proved in the same manner.

For any two idempotents p and q , we replace b and c by p and $1 - q$ respectively in Theorem 3.13 and 3.14, we obtain the following corollary.

Corollary 3.16. [11, Theorem 3.3] *Let $p, q \in E(R)$ and let $a \in R$ be such that a^\times exists. Then for $d \in R$ the following statements are equivalent:*

- (i) d^\times exists.
- (ii) $1 - p + a^\times dp$ is invertible.
- (iii) $q + (1 - q)da^\times$ is invertible.

4. CHARACTERIZATIONS OF ELEMENTS WITH EQUAL (b, c) -IDEMPOTENTS

Let $a^{\parallel(b,c)}$ exists. Since $a^{\parallel(b,c)}$ is an outer inverse of a , when it exists, then both $a^{\parallel(b,c)}a$ and $aa^{\parallel(b,c)}$ are idempotents. These will be referred to as the (b, c) -idempotents associated with a . We are interested in finding characterizations of those elements in the ring with equal (b, c) -idempotents.

In what follows, we will give necessary and sufficient conditions for $aa^{\parallel(b,c)} = dd^{\parallel(b,c)}$. We firstly establish an auxiliary result.

Lemma 4.1. *Let $a, b, c, d \in R$ such that $a^{\parallel(b,c)}$ and $d^{\parallel(b,c)}$ exist. Let $e = bb^-$ and $f = c^-c$, where b^- and c^- are fixed, but arbitrary inner inverses of b and c , respectively. Then*

- (i) $d^{\parallel(b,c)} = d^{\parallel(b,c)}aa^{\parallel(b,c)} = a^{\parallel(b,c)}ad^{\parallel(b,c)}.$
- (ii) $a^{\parallel(b,c)} = a^{\parallel(b,c)}dd^{\parallel(b,c)} = d^{\parallel(b,c)}da^{\parallel(b,c)}.$
- (iii) $e = ed^{\parallel(b,c)}aa^{\parallel(b,c)}de = ea^{\parallel(b,c)}ae = ed^{\parallel(b,c)}de.$
- (iv) $f = fda^{\parallel(b,c)}ad^{\parallel(b,c)}f = fdd^{\parallel(b,c)}f = faa^{\parallel(b,c)}f.$

Proof. (i). In view of (3.1) and (3.4), with the notation $e = bb^-$ and $f = c^-c$, we have

$$\begin{aligned} d^{\parallel(b,c)} &= (a^{\parallel(b,c)}de + 1 - e)^{-1}a^{\parallel(b,c)} = d^{\parallel(b,c)}aa^{\parallel(b,c)} \\ &= a^{\parallel(b,c)}(fda^{\parallel(b,c)} + 1 - f)^{-1} = a^{\parallel(b,c)}ad^{\parallel(b,c)}. \end{aligned}$$

(ii). We get these equalities by interchanging the roles of $a^{\parallel(b,c)}$ and $d^{\parallel(b,c)}$ in previous results.

(iii). By the Definition 1.1, we have $b = d^{\parallel(b,c)}db$. Multiplying on the right by b^- gives $e = d^{\parallel(b,c)}de$. Similarly, $e = ea^{\parallel(b,c)}ae$. Multiplying (i) on the right by de leads to $e = ed^{\parallel(b,c)}aa^{\parallel(b,c)}de$.

(iv). By the definition 1.1, we have $c = cad^{\parallel(b,c)}$ and, multiplying on the left by c^- , we get $f = fdd^{\parallel(b,c)}$. Similarly, $f = faa^{\parallel(b,c)}f$. Multiplying (ii) on the left by fd , one can see $f = fda^{\parallel(b,c)}ad^{\parallel(b,c)}f$. \square

Theorem 4.2. *Let $a, b, c, d \in R$ such that $a^{\parallel(b,c)}$ and $d^{\parallel(b,c)}$ exist. Then the following statements are equivalent:*

- (i) $aa^{\parallel(b,c)} = dd^{\parallel(b,c)}.$
- (ii) $aa^{\parallel(b,c)}dd^{\parallel(b,c)} = dd^{\parallel(b,c)}aa^{\parallel(b,c)}.$
- (iii) $ad^{\parallel(b,c)}da^{\parallel(b,c)} = da^{\parallel(b,c)}ad^{\parallel(b,c)}.$

(iv) $ad^{\|(b,c)} \in R^\#$ and $(ad^{\|(b,c)})^\# = da^{\|(b,c)}$.

(v) $da^{\|(b,c)} \in R^\#$ and $(da^{\|(b,c)})^\# = ad^{\|(b,c)}$.

Proof. (i) \Leftrightarrow (ii) \Leftrightarrow (iii). From Lemma 4.1 we obtain

$$(4.1) \quad \begin{aligned} aa^{\|(b,c)} &= aa^{\|(b,c)} dd^{\|(b,c)} = ad^{\|(b,c)} da^{\|(b,c)}; \\ dd^{\|(b,c)} &= dd^{\|(b,c)} aa^{\|(b,c)} = da^{\|(b,c)} ad^{\|(b,c)}. \end{aligned}$$

This leads to

$$\begin{aligned} aa^{\|(b,c)} = dd^{\|(b,c)} &\Leftrightarrow aa^{\|(b,c)} dd^{\|(b,c)} = dd^{\|(b,c)} aa^{\|(b,c)} \\ &\Leftrightarrow ad^{\|(b,c)} da^{\|(b,c)} = da^{\|(b,c)} ad^{\|(b,c)}. \end{aligned}$$

(iii) \Leftrightarrow (iv). Set $x = da^{\|(b,c)}$. We will prove that x is the group inverse of $ad^{\|(b,c)}$.

Combining (iii) with Lemma 4.1, we get

$$\begin{aligned} xad^{\|(b,c)} &= da^{\|(b,c)} ad^{\|(b,c)} = ad^{\|(b,c)} da^{\|(b,c)} = ad^{\|(b,c)} x; \\ ad^{\|(b,c)} xad^{\|(b,c)} &= a(d^{\|(b,c)} da^{\|(b,c)}) ad^{\|(b,c)} = a(a^{\|(b,c)} ad^{\|(b,c)}) = ad^{\|(b,c)}; \\ xad^{\|(b,c)} x &= xad^{\|(b,c)} da^{\|(b,c)} = xaa^{\|(b,c)} = da^{\|(b,c)} aa^{\|(b,c)} = x. \end{aligned}$$

This implies that $ad^{\|(b,c)} \in R^\#$ and $(ad^{\|(b,c)})^\# = da^{\|(b,c)}$. Conversely, if the latter holds, then $da^{\|(b,c)} ad^{\|(b,c)} = ad^{\|(b,c)} da^{\|(b,c)}$.

(iii) \Leftrightarrow (v). The proof is similar to the previous equivalence. \square

We state the result in terms of the other (b, c) -idempotent.

Theorem 4.3. *Let $a, b, c, d \in R$ such that $a^{\|(b,c)}$ and $d^{\|(b,c)}$ exist. Then the following statements are equivalent:*

- (i) $a^{\|(b,c)} a = d^{\|(b,c)} d$.
- (ii) $d^{\|(b,c)} da^{\|(b,c)} a = a^{\|(b,c)} ad^{\|(b,c)} d$.
- (iii) $a^{\|(b,c)} dd^{\|(b,c)} a = d^{\|(b,c)} aa^{\|(b,c)} d$.
- (iv) $a^{\|(b,c)} d \in R^\#$ and $(a^{\|(b,c)} d)^\# = d^{\|(b,c)} a$.
- (v) $d^{\|(b,c)} a \in R^\#$ and $(d^{\|(b,c)} a)^\# = a^{\|(b,c)} d$.

Next, we consider conditions under which the reverse order rule for the (b, c) -inverse of the product ad , $(ad)^{\|(b,c)} = d^{\|(b,c)} a^{\|(b,c)}$ holds.

Theorem 4.4. *Let $a, b, c, d \in R$ such that $a^{\parallel(b,c)}$ and $d^{\parallel(b,c)}$ exist. Then the following statements are equivalent:*

- (i) *ad has a (b, c) -inverse of the form $(ad)^{\parallel(b,c)} = d^{\parallel(b,c)}a^{\parallel(b,c)}$.*
- (ii) *$d^{\parallel(b,c)} = d^{\parallel(b,c)}add^{\parallel(b,c)}a^{\parallel(b,c)} = d^{\parallel(b,c)}a^{\parallel(b,c)}add^{\parallel(b,c)}$.*
- (iii) *$a^{\parallel(b,c)} = a^{\parallel(b,c)}add^{\parallel(b,c)}a^{\parallel(b,c)} = d^{\parallel(b,c)}a^{\parallel(b,c)}ada^{\parallel(b,c)}$.*

Proof. (i) \Leftrightarrow (ii). We first assume that ad has a (b, c) -inverse given by $(ad)^{\parallel(b,c)} = d^{\parallel(b,c)}a^{\parallel(b,c)}$. Then Lemma 4.1 is true for $(ad)^{\parallel(b,c)}$ in place of $a^{\parallel(b,c)}$. It follows that

$$d^{\parallel(b,c)} = d^{\parallel(b,c)}ad(ad)^{\parallel(b,c)} = (ad)^{\parallel(b,c)}add^{\parallel(b,c)}.$$

Substituting $(ad)^{\parallel(b,c)} = d^{\parallel(b,c)}a^{\parallel(b,c)}$ yields

$$d^{\parallel(b,c)} = d^{\parallel(b,c)}add^{\parallel(b,c)}a^{\parallel(b,c)} = d^{\parallel(b,c)}a^{\parallel(b,c)}add^{\parallel(b,c)}.$$

Conversely, if the latter identities hold then $y = d^{\parallel(b,c)}a^{\parallel(b,c)}$ is the (b, c) -inverse of ad . Indeed, since $d^{\parallel(b,c)}db = b$ and $c = cdd^{\parallel(b,c)}$, we have

$$\begin{aligned} yady &= d^{\parallel(b,c)}a^{\parallel(b,c)}add^{\parallel(b,c)}a^{\parallel(b,c)} = d^{\parallel(b,c)}a^{\parallel(b,c)}; \\ yadb &= d^{\parallel(b,c)}a^{\parallel(b,c)}adb = d^{\parallel(b,c)}a^{\parallel(b,c)}add^{\parallel(b,c)}db = d^{\parallel(b,c)}db = b; \\ cady &= cadd^{\parallel(b,c)}a^{\parallel(b,c)} = cdd^{\parallel(b,c)}add^{\parallel(b,c)}a^{\parallel(b,c)} = cdd^{\parallel(b,c)} = c. \end{aligned}$$

(ii) \Rightarrow (iii). By Lemma 4.1 we have $a^{\parallel(b,c)} = a^{\parallel(b,c)}dd^{\parallel(b,c)} = d^{\parallel(b,c)}da^{\parallel(b,c)}$. By (ii), one can see

$$a^{\parallel(b,c)} = a^{\parallel(b,c)}d(d^{\parallel(b,c)}add^{\parallel(b,c)}a^{\parallel(b,c)}) = (d^{\parallel(b,c)}a^{\parallel(b,c)}add^{\parallel(b,c)})da^{\parallel(b,c)}.$$

Hence, it is easy to get $a^{\parallel(b,c)} = a^{\parallel(b,c)}add^{\parallel(b,c)}a^{\parallel(b,c)} = d^{\parallel(b,c)}a^{\parallel(b,c)}ada^{\parallel(b,c)}$.

(iii) \Rightarrow (ii). The proof is similar to (i) \Rightarrow (iii). □

Theorem 4.5. *Let $a, b, c, d \in R$ such that $a^{\parallel(b,c)}$ and $d^{\parallel(b,c)}$ exist. Then the following statements are equivalent:*

- (i) *$a^{\parallel(b,c)}a = dd^{\parallel(b,c)}$.*
- (ii) *$a^{\parallel(b,c)}dd^{\parallel(b,c)}a = dd^{\parallel(b,c)}aa^{\parallel(b,c)}$.*
- (iii) *$d^{\parallel(b,c)}da^{\parallel(b,c)}a = da^{\parallel(b,c)}ad^{\parallel(b,c)}$.*
- (iv) *$a^{\parallel(b,c)} = dd^{\parallel(b,c)}a^{\parallel(b,c)}$ and $d^{\parallel(b,c)} = d^{\parallel(b,c)}a^{\parallel(b,c)}a$.*
- (v) *$a^{\parallel(b,c)}ad^{\parallel(b,c)} = d^{\parallel(b,c)}a^{\parallel(b,c)}a$ and $a^{\parallel(b,c)}dd^{\parallel(b,c)} = dd^{\parallel(b,c)}a^{\parallel(b,c)}$.*

If any of the previous statements is valid, then $(ad)^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}$.

Proof. (i) \Leftrightarrow (ii) \Leftrightarrow (iii). From Lemma 4.1 we obtain

$$(4.2) \quad \begin{aligned} a^{\|(b,c)}a &= a^{\|(b,c)}dd^{\|(b,c)}a = d^{\|(b,c)}da^{\|(b,c)}a; \\ dd^{\|(b,c)} &= dd^{\|(b,c)}aa^{\|(b,c)} = da^{\|(b,c)}ad^{\|(b,c)}. \end{aligned}$$

Hence, it gives that

$$\begin{aligned} a^{\|(b,c)}a = dd^{\|(b,c)} &\Leftrightarrow a^{\|(b,c)}dd^{\|(b,c)}a = dd^{\|(b,c)}aa^{\|(b,c)} \\ &\Leftrightarrow d^{\|(b,c)}da^{\|(b,c)}a = da^{\|(b,c)}ad^{\|(b,c)}. \end{aligned}$$

(i) \Leftrightarrow (iv). The necessary condition is immediate. Next, we assume that $a^{\|(b,c)} = dd^{\|(b,c)}a^{\|(b,c)}$ and $d^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}a$. Then we have $a^{\|(b,c)}a = dd^{\|(b,c)}a^{\|(b,c)}a$ and $dd^{\|(b,c)} = dd^{\|(b,c)}a^{\|(b,c)}a$. So $a^{\|(b,c)}a = dd^{\|(b,c)}$, as desired.

(v) \Leftrightarrow (i). The proof is similar to the above.

Finally, we will prove that $dd^{\|(b,c)} = a^{\|(b,c)}a$ implies that $(ad)^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}$. Since $d^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}a$, we have $d^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}add^{\|(b,c)}$. Moreover, since $d^{\|(b,c)} = d^{\|(b,c)}aa^{\|(b,c)}$ by Lemma 4.1, using $dd^{\|(b,c)} = a^{\|(b,c)}a$, it follows that

$$d^{\|(b,c)} = d^{\|(b,c)}aa^{\|(b,c)} = d^{\|(b,c)}aa^{\|(b,c)}aa^{\|(b,c)} = d^{\|(b,c)}add^{\|(b,c)}a^{\|(b,c)}.$$

By Theorem 4.4 our assertion is proved. \square

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